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or, in expanded form,

$$(3) \quad \frac{df(X)}{da} = \frac{2X^5 + 6aX^4 + 3(1-b)X^3 + b(1-b)X + 2ab - 2bX}{X-a}.$$

The roots of  $g(x) = 0$  satisfy the relations  $0 < x_1 < a < x_2 < 2a$ , from which we can show that the numerator on the right side of (3) is positive for both values of  $X$ . In fact, all the terms are positive except the last; but if  $X = x_1 < a$ , we have  $2ab - 2bX > 0$ , so that the numerator is positive in this case, while if  $X = x_2$  we have, since  $b < a^2$ , and  $a < x_2 < 2a$ ,

$$\begin{aligned} 2X^5 + b(1-b)X - 2bX + 2ab &= (2X^5 - b^2X) + (2ab - bX), \\ &> (2a^4 - a^4)X + b(2a - 2a) > 0. \end{aligned}$$

Since  $x_1 - a$  is negative and  $x_2 - a$  is positive, we thus have

$$\frac{df(x_1)}{da} < 0, \quad \frac{df(x_2)}{da} > 0.$$

It follows that  $f(x_1)$  has its greatest value, and  $f(x_2)$  its least value, when  $a = \sqrt{b}$ , in which case  $x_1 = x_2 = a$ , and  $f(x_1) = f(x_2) = 0$ . The relations (1) follow at once.

II. We will show that the positive root of  $f(x) = 0$  (there can be only one, by Descartes' Rule) lies between  $x = \sqrt{b}$  and  $x = a$ . The interval  $[\sqrt{b}, a]$  is, however, wholly comprised between the two roots of  $g(x) = 0$ , since  $g(0)$  and  $g(\infty)$  are positive, while

$$g(\sqrt{b}) = 2b - 2a\sqrt{b} = 2\sqrt{b}(\sqrt{b} - a)$$

is negative, and  $g(a) = b - a^2$  is also negative.

We now prove that  $f(\sqrt{b}) < 0$  and  $f(a) > 0$ . In the first place,

$$f(\sqrt{b}) = 2ab^2 + 2b\sqrt{b}(1-b) - 2ab.$$

Since  $\sqrt{b} < a$ , the substitution of  $a$  for  $\sqrt{b}$  in the second term above gives

$$f\sqrt{b} < 2ab^2 + 2ab(1-b) - 2ab = 0.$$

On the other hand, we have

$$f(a) = 2a^5 + a^3(1-b) + ab(1-b) - 2ab = 2a^5 + a^3 - ab(a^2 + 1 + b).$$

If we replace each  $b$  above by  $a^2$  we obtain the inequality

$$f(a) > 2a^5 + a^3 - a^3(a^2 + 1 + a^2) = 0.$$

Also solved by E. J. MOULTON, J. E. ROWE, H. H. CONWELL, J. A. BULLARD, J. L. RILEY, HORACE OLSON, J. W. BALDWIN, and the PROPOSER.

#### 470. Proposed by ERNEST W. BROWN, Yale University.

There are  $n$  numbers each lying between  $-\frac{1}{2}$  and  $+\frac{1}{2}$ , such that any value of each between these limits is equally probable. What is the probability that their sum will lie between  $s - \frac{1}{2}$  and  $s + \frac{1}{2}$ , where  $s$  is an integral multiple of  $\frac{1}{2}$ .

#### SOLUTION BY C. F. GUMMER, Kingston, Ontario.

Consider the more general problem where the  $n$  numbers are chosen at random from the intervals  $(a_1, b_1), \dots, (a_n, b_n)$ , and the sum is to lie in the interval  $(a, b)$ .

Let  $c_1 = b_1 - a_1, \dots, c_n = b_n - a_n$ .

Let  $f_n(x) \cdot dx$  be the probability that the sum of the first  $r$  numbers will lie between  $x$  and  $x + dx$ .

Then  $f_1(x) = 1/c_1$  when  $a_1 < x < b_1$ , and zero in other cases, and

$$f_r(x) = \int_{x-b_r}^{x-a_r} \frac{f_{r-1}(\xi) \cdot d\xi}{c_r}.$$

To evaluate  $f_n(x)$ , let us use the notations

$$\phi(x) = 1 \text{ for } x > 0, \quad \phi(x) = -1 \text{ for } x < 0, \quad \phi_r(x) = x^r \phi(x).$$

The function  $\phi(x)$  is discontinuous but integrable. Since  $\int_0^a \phi(x) dx = \phi_1(\alpha)$ , therefore,  $\int_a^\beta \phi(x) dx = \phi_1(\beta) - \phi_1(\alpha)$ . In like manner,

$$\int_a^\beta \phi_r(x) dx = \frac{\phi_{r+1}(\beta) - \phi_{r+1}(\alpha)}{r+1}.$$

Now

$$f_1(x) = \frac{1}{2c_1} \{ \phi(x - a_1) - \phi(x - b_1) \}.$$

Hence,

$$\begin{aligned} f_2(x) &= \frac{1}{c_2} \int_{x-b_2}^{x-a_2} f_1(\xi) d\xi \\ &= \frac{1}{2c_1 c_2} \{ \phi_1(x - a_1 - a_2) - \phi_1(x - a_1 - b_2) - \phi_1(x - a_2 - b_1) + \phi_1(x - b_1 - b_2) \}. \end{aligned}$$

We thus obtain by induction

$$f_n(x) = \frac{1}{2c_1 \cdots c_n} \frac{\Sigma (-1)^k \phi_{n-1}(x - \alpha_1 - \beta_2 - \cdots - \kappa_n)}{n-1},$$

where, in the course of the summation, each of the letters  $\alpha, \beta, \dots, k$  is taken to represent either  $a$  or  $b$  independently of the others, and  $k$  is the number of them standing for  $b$  in any term.

The probability that the sum lies between  $a$  and  $b$  is then

$$(1) \quad \int_a^b f_n(x) dx = \frac{1}{2c_1 \cdots c_n} \frac{\Sigma (-1)^{k'-1} \phi_n(\omega - \alpha_1 - \cdots - \kappa_n)}{n},$$

where  $\omega$  stands for  $a$  or  $b$ , and if it is  $b$  it is counted in  $k'$ .

We may modify the formula (1) as follows: If  $a$  is greater than  $b_1 + b_2 + \cdots + b_n$  ( $b > a$ ,  $b_r > a_r$ ), the probability is zero. But the  $\phi_n$ 's now reduce to  $n$ th powers. Hence,

$$\frac{1}{2c_1 c_2 \cdots c_n} \frac{\Sigma (-1)^{k'-1} (\omega - \alpha_1 - \cdots - \kappa_n)^n}{n} = 0$$

for all values of  $a$  and  $b$  beyond certain limits, and is therefore identically zero. On adding this zero quantity to the right-hand side of (1) we cause all the  $\phi_n$ 's which are not mere  $n$ th powers to cancel, and double the others.

Hence, the probability is

$$\frac{1}{n c_1 c_2 \cdots c_n} \Sigma (-1)^{k'-1} (\omega - \alpha_1 - \cdots - \kappa_n)^n,$$

the summation covering all the cases where  $\omega - \alpha_1 - \cdots - \kappa_n$  is positive.

It is easy to deduce for the special case propounded that the probability is

$$\frac{1}{n} \left\{ \left( s + \frac{n+1}{2} \right)^n - n+1 C_1 \left( s + \frac{n-1}{2} \right)^n + \cdots + (-1)^r n+1 C_r \left( s + \frac{n+1}{2} - r \right)^n \right\},$$

where  $r$  is the next integer below  $s + (n+1)/2$ , and  $s$  may be any real number.

## GEOMETRY.

Solutions of 497 were received from J. B. Reynolds, Elijah Swift, R. M. Mathews, and H. R. Howard after selections for publication were made.

### 498. Proposed by FRANK R. MORRIS, Glendale, California.

To trisect an angle  $ABC$ , on  $BA$  and  $BC$  take  $D$  and  $E$  equidistant from  $B$ . Using  $DE$  as a diameter draw the semicircle  $DFGE$ . With the same radius and  $D$  and  $E$  as centers draw arcs locating the points  $F$  and  $G$  on this semicircle. Connect  $F$  and  $G$  with  $B$ . Prove that this method trisects a right angle and a straight angle and that it does not trisect an oblique angle.